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A SEMI-LAGRANGIAN METHOD BASED ON WENO INTERPOLATION

Dokkyun Yi* and Hyunsook Kim**

Abstract. In this paper, a general Weighted Essentially Non-Oscillatory (WENO) interpolation is proposed and applied to a semi-Lagrangian method. The proposed method is based on the conservation law, and characteristic curves are used to complete the semi-Lagrangian method. Therefore, the proposed method satisfies conservation of mass and is free of the CFL condition which is a necessary condition for convergence.Using a several standard examples, the proposed method is compared with the third order Strong Stability Preserving (SSP) Runge-Kutta method to verify the high-order accuracy.

1. Introduction

The Weighted Essentially Non-Oscillatory (WENO) interpolations [8, 9, 10, 12, 13] are well known effective high-order non-oscillatory schemes based on Eulerian approach methods (grid based methods) as Finite Volume methods (FVM) or Finite Difference methods (FDM). However, one of the drawbacks is that the methods are limited by the Courant-Friedrichs-Levy (CFL) condition which is a necessary condition for convergence.

In order to resolve this challenging problem, a semi-Lagrangian method constructed by characteristic curves and grid points is introduced [1, 2, 3, 4, 5, 14]. Our main idea is that time integration of a given conservation law (a partial differential eqnarray to be solved) is changed into space integration by using characteristic curves as the semi-Lagrangian method, and the Gaussian quadrature rule to obtain high-order accuracy. Therefore, it can not be guaranteed that the points of Gaussian quadrature rule computed by the characteristic curves are on regula grid points

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Correspondence should be addressed to Hyunsook Kim, Kim.HS@daegu.ac.kr.

(evaluating points or Eulerian points) [15]. In order to proceed with a semi-Lagrangian, we should compute the value at the points of Gaussian quadrature rule. Unfortunately, since the existing WENO interpolation is constructed at regular grids, it can not apply to a semi-Lagrangian method. This paper introduces a general WENO interpolation to apply to a semi-Lagrangian method. The general WENO interpolation is constructed by using the idea of the existing WENO interpolation. In particular, we can confirm that the proposed methodology is well applied to the conservation law, and is excellent by comparing with the numerical results of the existing WENO interpolation computed on the third order Strong Stability Preserving (SSP) Runge-Kutta method [6, 7, 11].

2. Concept of the proposed method

2.1. Scalar conservation laws

We consider numerical solutions of the scalar conservation law

$$u_t + \nabla \cdot \mathbf{F}(u) = 0, \qquad u(\mathbf{x}, 0) = u_0(\mathbf{x}), \tag{2.1}$$

where $u_0(\mathbf{x})$ is assumed to be a bounded variation function. For convenience, the one-dimensional version,

$$u_t + f(u)_x = 0, \qquad u(x, 0) = u_0(x), \tag{2.2}$$

is used in this paper to illustrate the main ideas. The main difficulty in solving (2.1) is that the solution may contain discontinuities, even if the initial condition is smooth.

To solve (2.2), we discretize space and time by assuming uniform mesh spacing of $h = \Delta x$ and Δt, respectively. $x_i = i \times h$ $(i = 0, \cdots, N)$ denotes the spatial grid points; $t^{n+1} = t^n + \Delta t$ the time stages; $I_i = [x_{i-1/2}, x_{i+1/2}]$, the cells; and $\bar{u}_i^n = \frac{1}{h}\int_{x_{i-1/2}}^{x_{i+1/2}} u(x, t^n)dx$ $(i = 0, \cdots, N)$, the cell average of cell I_i at time t^n, where $u_i^n = u(x_i, t^n)$ $(i = 0, \cdots, N)$ respects the point value. Integrating (2.2) over $I_i \times [t^n, t^{n+1}]$ gives

$$\int_t^{t+\Delta t}\int_{I_i} u_t \; dxdt = -\int_t^{t+\Delta t}\int_{I_i} f(u)_x \; dxdt.$$

From Fubini's theorem, we get

$$\int_{I_i}\int_t^{t+\Delta t} u_t \; dtdx = -\int_t^{t+\Delta t}\int_{I_i} f(u)_x \; dxdt.$$

This implies that

$$\int_{I_i} u(t+\Delta t) - u(t)\ dx = -\int_t^{t+\Delta t} f(u(x_{i+\frac{1}{2}}))) - f(u(x_{i-\frac{1}{2}}))\ dxdt.$$

Finally, our target eqnarray

$$(2.3)\ \bar{u}_i^{n+1} = \bar{u}_i^n - \frac{1}{h}\int_{t^n}^{t^{n+1}} \left[f(u(x_{i+1/2},t)) - f(u(x_{i-1/2},t))\right] dt$$

is obtained. This eqnarray describes the cell average advance from t^n to t^{n+1}. To descretize (2.3), it is necessary to design a time integration scheme.

2.2. Schemes for time accuracy and quadrature integration

Eulerian approaches have the advantage of high-order accuracy, which reduces the computational cost and yields similar or better results than low-order methods on a fine mesh. However, to achieve the high-order accuracy, we consider two factors, such as space and time. A finite volume WENO scheme is one of the most powerful schemes for the spatial accuracy, which provides non-oscillatory high-order accuracy. It uses semi-discretization of (2.2), which is the integration form over the interval I_i,

$$(2.4)\qquad \frac{d\bar{u}(x_i,t)}{dt} = -\frac{1}{h}\left(f(u(x_{i+1/2},t)) - f(u(x_{i-1/2},t))\right).$$

To achieve the high-order time accuracy of the form (2.4), Runge-Kutta-type time discretizations [12, 7, 6, 11] have been widely employed. However, such methods require a repetitive single step or a multi-step computation, resulting in a high computation cost. For example, the third order SSP Runge-Kutta method (with the CFL coefficient $c = 1$) [12] is given by

$$\begin{aligned} u^{(1)} &= u^n + \Delta t F(u^n), \\ u^{(2)} &= \frac{3}{4}u^n + \frac{1}{4}\left(u^{(1)} + \Delta F(u^{(1)})\right), \\ u^{n+1} &= \frac{1}{3}u^n + \frac{2}{3}\left(u^{(2)} + \Delta F(u^{(2)})\right), \end{aligned}$$

where $F(u)$ is the spatial operator, and the third order SSP multi-step method [11] (with the CFL coefficient $c = 1/3$) is given by

$$u^{n+1} = \frac{16}{27}\left(u^n + 3\Delta F(u^n)\right) + \frac{11}{26}\left(u^{n-3} + \frac{12}{11}\Delta F(u^{n-3})\right).$$

Our numerical scheme (2.3) has the advantage of high-order time accuracy because our time accuracy is determined by the integration form. We use the Gaussian quadrature rule to obtain an accurate integration value. A quadrature rule is an approximation of the definite integral of a function, usually stated as the weighted sum of the function values at specified points within the domain of integration.

2.3. Time integration scheme of proposed method

The idea of the proposed method is to use the characteristic curve to approximate the time integration. The Gaussian quadrature integration form of (2.3) is

$$\bar{u}_i^{n+1} = \bar{u}_i^n - \frac{\Delta t}{h}\sum_l \frac{\omega_l}{2}[f\left(u^n(x_{i+1/2}, \frac{t^n+t^{n+1}}{2}+\frac{\Delta t}{2}\tau_l)\right) - f\left(u^n(x_{i-1/2}, \frac{t^n+t^{n+1}}{2}+\frac{\Delta t}{2}\tau_l)\right)]dt,$$

where ω_l denotes the quadrature weights and τ_l denotes the nodes of the quadrature. To calculate the integration form, we use the characteristics $X(t) = (x_0, t)$ satisfying

$$\frac{dX}{dt} = f'(u), \tag{2.5}$$

which is the fundamental theory of the semi-Lagrangian method for determining the departure point. Let $f'(u) = v$ and x at time t be the point that becomes $\tilde{x}$ at time $t + \Delta t$ using characteristics; then,

$$x = \tilde{x} + \int_t^{t+\Delta t} v dt. \tag{2.6}$$

In particular, for a linear advection eqnarray $f'(u) = a$, where a is constant, (2.6) becomes $x = \tilde{x} + a\Delta t$. Assuming $v \neq 0$; and since $dX = vdt$,

$$\int_t^{t+\Delta t} f dt = \int_t^{t+\Delta t} \frac{f}{v} v dt = \int_{\tilde{x}}^{x} \frac{f}{v} dX. \tag{2.7}$$

From (2.7), the computable form $\int_{\tilde{x}}^{x} f/v \; dX$ is obtained from the incomputable form $\int_t^{t+\Delta t} f dt$. Let $g(u) = f/v$; then, our objective is to approximate

$$\int_{\tilde{x}}^{x} g(u) ds. \tag{2.8}$$

From (2.8), the conservative form (2.3) becomes

$$\bar{u}(t+\Delta t) = \bar{u}(t) - \int_{\tilde{x}_{i+\frac{1}{2}}}^{x_{i+\frac{1}{2}}} g(u)ds + \int_{\tilde{x}_{i-\frac{1}{2}}}^{x_{i-\frac{1}{2}}} g(u)ds. \tag{2.9}$$

Finally, our scheme is constructed using the global Lax-Friedrich scheme and the quadrature points integration, i.e., the numerical flux function in (2.9) is approximated by

$$\begin{aligned}\int_{\tilde{x}_{i+\frac{1}{2}}}^{x_{i+\frac{1}{2}}} g(u)ds &\approx h(\hat{u}^-_{i+\frac{1}{2}}, \hat{u}^+_{i+\frac{1}{2}}) \\ &= \frac{1}{2}\Big(\sum_l \frac{\omega_l}{2}\Big[v\left(g(p^-(v_c+v\tau_l)) + g(p^+(v_c+v\tau_l))\right) \\ &\qquad - \max|f'(u^n)|\big(p^+(v_c+v\tau_l) - p^-(v_c+v\tau_l)\big)\Big]\Big),\end{aligned}$$

where $v = (x_{i+1/2} - \tilde{x}_{i+1/2})/2$, $v_c = (x_{i+1/2} + \tilde{x}_{i+1/2})/2$, ω_l and τ_l are weights and points of Gaussian quadrature, respectively, and p^- and p^+ are reconstruction polynomials obtained from $\bar{u}^n$.

2.4. Reconstruction and approximation in 1D

In this subsection, the problems of interpolation and approximation for obtaining polynomials $p(x)$ are presented in one dimension. In other words, a polynomial $p_i(x)$ of degree at most $k-1$ is presented, and the polynomial $p_i(x)$ is the k-th order accurate approximation of the function $u(x)$ inside I_i. In particular, this gives approximations of the functions $u(x)$ in the cell

$$u^-_{i+\alpha} = p_i(x_{i+\alpha}), \quad u^+_{i-\alpha} = p_i(x_{i-\alpha}), \quad i = 1, ..., N, \quad 0 < \alpha < 1.$$

Since the mappings from the given cell averages $\bar{u}_j$ in the stencil $S(i) \equiv \{I_{i-r}, ..., I_{i+s}\}$ to the values $u^-_{i+\alpha}$ and $u^+_{i-\alpha}$ are linear, there exist constants c_{rj} and $\tilde{c}_{rj}$ that depend on the left shift of the stencil r of the stencil $S(i)$, but not on the function u itself, such that

$$u^-_{i+\alpha} = \sum_{j=0}^{k-1} c_{rj}\bar{u}_{i-r+j}, \quad u^+_{i-\alpha} = \sum_{j=0}^{k-1} \tilde{c}_{rj}\bar{u}_{i-r+j}.$$

We note that the difference between the values with superscripts $\pm$ at the same location $x_{i+\alpha}$ is due to the possibility of using difference stencils for cell I_i and cell I_{i+1}. Using stencil $S(i)$ to approximate $x_{i+\alpha}$, we can

drop the superscripts $\pm$ and eliminate the need to consider $\tilde{c}_{rj}$ because it is clear that

$$\tilde{c}_{rj} = c_{r-1,j}.$$

To understand how the constants $\{c_{rj}\}$ are obtained, the primitive function of $u(x)$,

$$U(x) \equiv \int_{-\infty}^{x} u(\xi)d\xi,$$

is introduced, where the lower limit $-\infty$ can be replaced by any fixed number. Let $U(x)$ be interpolated by $P(x)$ which is a unique polynomial of degree at most k, and Let $p(x)$ be the derivative of $P(x)$; then, the Lagrange form of the interpolation polynomial $P(x)$

$$P(x) = \sum_{m=0}^{k} U(x_{i-r+m-\frac{1}{2}}) \prod_{\substack{l=0 \\ l\neq m}} \frac{x - x_{i-r+l-\frac{1}{2}}}{x_{i-r+m-\frac{1}{2}} - x_{i-r+l-\frac{1}{2}}}$$

is obtained. Taking the derivative of $P(x) - U(x_{i-r-\frac{1}{2}})$, we obtain

$$p(x) = \sum_{m=0}^{k} \sum_{j=0}^{m-1} \bar{u}_{i-r+j} \Delta x_{i-r+j} \left(\frac{\displaystyle\sum_{\substack{l=0 \\ l\neq m}}^{m} \prod_{\substack{q=0 \\ q\neq m,l}}^{k} (x - x_{i-r+q-\frac{1}{2}})}{\displaystyle\prod_{\substack{l=0 \\ l\neq m}}^{k} (x_{i-r+m-\frac{1}{2}} - x_{i-r+l-\frac{1}{2}})} \right).$$

Let $x = x_{i+\alpha}$; then, for a uniform grid, $\Delta x_i = \Delta x$; thus, the expression for c_{rj} does not depend on i or Δx:

$$c_{rj} = \sum_{m=j+1}^{k} \frac{\displaystyle\sum_{\substack{l=0 \\ l\neq m}}^{m} \prod_{\substack{q=0 \\ q\neq m,l}}^{k} (r - q + \alpha + \frac{1}{2})}{\displaystyle\prod_{\substack{l=0 \\ l\neq m}}^{k} (m - l)}.$$

2.5. Reconstruction of point values by WENO

The basic idea of WENO is as follows: instead of using only one of the candidate stencils to obtain the reconstruction, one uses a convex

k	r	$j=0$	$j=1$	$j=2$	$j=3$
2	-1	$2-\alpha$	$\alpha-1$		
	0	$1-\alpha$	α		
	1	$-\alpha$	$\alpha+1$		
3	-1	$\frac{\alpha^2}{2}-\frac{5\alpha}{2}+\frac{71}{24}$	$-\alpha^2+4\alpha-\frac{35}{12}$	$\frac{\alpha^2}{2}-\frac{3\alpha}{2}+\frac{23}{24}$	
	0	$\frac{\alpha^2}{2}-\frac{3\alpha}{2}+\frac{23}{24}$	$-\alpha^2+2\alpha+\frac{1}{12}$	$\frac{\alpha^2}{2}-\frac{\alpha}{2}-\frac{1}{24}$	
	1	$\frac{\alpha^2}{2}-\frac{\alpha}{2}-\frac{1}{24}$	$\frac{13}{12}-\alpha^2$	$\frac{\alpha^2}{2}+\frac{\alpha}{2}-\frac{1}{24}$	
	2	$\frac{\alpha^2}{2}+\frac{\alpha}{2}-\frac{1}{24}$	$-\alpha^2-2\alpha+\frac{1}{12}$	$\frac{\alpha^2}{2}+\frac{3\alpha}{2}+\frac{23}{24}$	
4	-1	$\frac{-((\alpha-3)(4\alpha^2-24\alpha+31))}{24}$	$\frac{\alpha^3}{2}-4\alpha^2+\frac{75\alpha}{8}-\frac{17}{3}$	$\frac{-\alpha^3}{2}+\frac{7\alpha^2}{2}-\frac{55\alpha}{8}+\frac{89}{24}$	$\frac{((\alpha-2)(4\alpha^2-16\alpha+11))}{24}$
	0	$\frac{-((\alpha-2)(4\alpha^2-16\alpha+11))}{24}$	$\frac{\alpha^3}{2}-\frac{5\alpha^2}{2}+\frac{23\alpha}{8}+\frac{5}{24}$	$\frac{-\alpha^3}{2}+2\alpha^2-\frac{11\alpha}{8}-\frac{1}{6}$	$\frac{-((\alpha-1)(-4\alpha^2+8\alpha+1))}{24}$
	1	$\frac{((\alpha-1)(-4\alpha^2+8\alpha+1))}{24}$	$\frac{\alpha^3}{2}-\alpha^2-\frac{5\alpha}{8}+\frac{13}{12}$	$\frac{-\alpha^3}{2}+\frac{\alpha^2}{2}+\frac{9\alpha}{8}-\frac{1}{24}$	$\frac{(\alpha(4\alpha^2-5))}{24}$
	2	$\frac{-(\alpha(4\alpha^2-5))}{24}$	$\frac{\alpha^3}{2}+\frac{\alpha^2}{2}-\frac{9\alpha}{8}-\frac{1}{24}$	$\frac{-\alpha^3}{2}-\alpha^2+\frac{5\alpha}{8}+\frac{13}{12}$	$\frac{((\alpha+1)(4\alpha^2+8\alpha-1))}{24}$
	3	$\frac{-((\alpha+1)(4\alpha^2+8\alpha-1))}{24}$	$\frac{\alpha^3}{2}+2\alpha^2+\frac{11\alpha}{8}-\frac{1}{6}$	$\frac{-\alpha^3}{2}-\frac{5\alpha^2}{2}-\frac{23\alpha}{8}+\frac{5}{24}$	$\frac{((\alpha+2)(4\alpha^2+16\alpha+11))}{24}$

TABLE 1. The constants c_{rj}

combination of all of them. To be more precise, suppose the k candidate stencils

$$S_r(i) = \{x_{i-r}, \dots, x_{i-r+k-1}\}, \quad r = 0, \dots, k-1 \tag{2.10}$$

produce k different reconstructions of the value v_α,

$$v_\alpha^{(r)} = \sum_{j=0}^{k-1} c_{rj} \bar{v}_{i-r+j}, \quad r = 0, \dots, k-1; \tag{2.11}$$

then, WENO reconstruction would involve a convex combination of all $v_\alpha^{(r)}$ defined in (2.11) as a new approximation;

$$v_\alpha = \sum_{r=0}^{k-1} \omega_r v_\alpha^{(r)}.$$

The key to the success of WENO is the choice of the weights ω_r, which should satisfy $\omega_r \geq 0$, $\sum_{r=0}^{k-1} \omega_r = 1$ for stability and consistency. If the function $v(x)$ is smooth in all the candidate stencils (2.10), there exist constants d_r such that

$$v_\alpha = \sum_{r=0}^{k-1} d_r v_\alpha^{(r)},$$

which are always positive, and owing to consistency, $\sum_{r=0}^{k-1} d_r = 1$. When the function $v(x)$ has a discontinuity in one or more of the stencils (2.10),

the corresponding weights ω_r are essentially 0. This leads to the form

$$\omega_r = \frac{\alpha_r}{\sum_{s=0}^{k-1} \alpha_s}, \quad r = 0, \ldots, k-1$$

with

$$\alpha_r = \frac{d_r}{(\epsilon + \beta_r)^2},$$

where ϵ is a small positive constant to avoid division by zero. Let the reconstruction polynomial on the stencil $S_r(i)$ be denoted by $p_r(x)$; then, the smoothness indicator β_r is defined as

$$\beta_r = \sum_{l=1}^{k-1} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \Delta x^{2l-1} \left(\frac{\partial^l p_r(x)}{\partial^l x} \right)^2 dx.$$

3. Numerical test

First, we present simple examples for obtaining the convergence order and the advantage of the spatial high-order methods.

Example 3.1 (one-dimensional linear translation). *The following linear scalar equation,*

$$u_t + u_x = 0,$$
$$u(x, 0) = u_0(x),$$

is solved with periodic boundary conditions.

The smooth initial condition, i.e., $u(x, 0) = \sin(x) + 1$, is used on $[0, 2\pi]$. We compute errors that are measured in L^1 and L^∞ norms at $t = 2$. For the purpose of showing time order accuracy, the time interval is taken $\Delta t = O(\Delta x)$, specifically $\Delta t = (8/25)\Delta x$. Table 2 lists the 5th order WENO convergence rates and numerical errors measured in L^1 and L^∞ norms that are computed using the one-dimensional linear advection equation in $[0, 2\pi]$. Table 3 lists the 7th order WENO convergence rates and numerical errors measured in L^1 and L^∞ norms that are computed using the one-dimensional linear advection equation in $[0, 2\pi]$. Table 4 lists the 9th order WENO convergence rates and numerical errors measured in L^1 and L^∞ norms that are computed using the one-dimensional linear advection equation in $[0, 2\pi]$. From Table 2, 3, and 4, the proposed method with high order WENO reconstructions obtains the expected convergence rates.

	SSP Runge-Kutta 3 with 5th order WENO			
Nx	L^1 error	Order	$L^\infty error$	Order
16	9.39E-03	-	2.61E-03	-
32	2.83E-04	5.05	9.65E-05	4.76
64	8.87E-06	5.00	3.00E-06	5.01
128	3.08E-07	4.85	9.94E-08	4.91
256	1.35E-08	4.51	4.01E-09	4.63
512	9.11E-10	3.89	2.41E-10	4.06
	The proposed method with 5th order WENO			
Nx	L^1 error	Order	$L^\infty error$	Order
16	9.06E-03	-	2.52E-03	-
32	2.68E-04	5.08	8.98E-05	4.81
64	8.15E-06	5.04	2.73E-06	5.04
128	2.53E-07	5.01	8.38E-08	5.02
256	7.91E-09	5.00	2.57E-09	5.03
512	2.47E-10	5.00	7.48E-11	5.10

TABLE 2

	SSP Runge Kutta 3 with 7th order WENO			
Nx	L^1 error	Order	$L^\infty error$	Order
16	7.80E-04	-	2.87E-04	-
32	9.07E-06	6.43	6.71E-06	5.42
64	3.36E-07	4.75	2.61E-07	4.69
128	4.17E-08	3.01	1.56E-08	4.06
256	5.21E-09	3.00	1.45E-09	3.43
512	6.51E-10	3.00	1.67E-10	3.11
	The proposed method with 7th order WENO			
Nx	L^1 error	Order	$L^\infty error$	Order
16	7.50E-04	-	2.82E-04	-
32	9.15E-06	6.36	5.86E-06	5.59
64	1.39E-07	6.04	1.67E-07	5.13
128	2.13E-09	6.03	5.00E-09	5.06
256	3.25E-11	6.03	1.37E-10	5.19
512	5.03E-13	6.02	3.84E-12	5.15

TABLE 3

EXAMPLE 3.2 (two-dimensional linear transport). *The two-dimensional linear advection equation,*

$$u_t + u_x + u_y = 0,$$
$$u(x, y, 0) = u_0(x, y),$$

is solved with periodic boundary conditions.

	SSP Runge Kutta 3 with 9th order WENO			
Nx	L^1 error	Order	$L^\infty error$	Order
16	3.08E-05	-	9.13E-06	-
32	2.68E-06	3.52	6.59E-07	3.79
64	3.33E-07	3.01	8.33E-08	2.98
	The proposed method with 9th order WENO			
Nx	L^1 error	Order	$L^\infty error$	Order
16	2.06E-05	-	5.09E-06	-
32	3.05E-08	9.40	1.02E-08	8.97
64	5.37E-11	9.15	1.86E-11	9.09

TABLE 4

	SSP Runge Kutta 3 with 9th order WENO			
Nx,Ny	L^1 error	Order	$L^\infty error$	Order
16	6.89E-04	-	3.41E-05	-
32	4.47E-05	3.95	2.18E-06	3.97
64	5.53E-06	3.01	2.79E-07	2.97
	The proposed method with 9th order WENO			
Nx,Ny	L^1 error	Order	$L^\infty error$	Order
9th order spatial accuracy				
16	4.46E-04	-	2.40E-05	-
32	7.01E-07	9.31	5.69E-08	8.72
64	1.34E-09	9.04	1.10E-10	9.01

TABLE 5

To check the convergence rates, the smooth initial condition $u(x, y, 0) = \sin(\pi x/3.5)\sin(\pi y/3.5)$ is used on $[0, 7] \times [0, 7]$. For a two-dimensional problem, we use the Strang splitting method, originally proposed in [2]. Table 5 lists the L^1 and L^∞ errors and convergence rates of the schemes applied to a two-dimensional linear advection equation with smooth initial data $u_0(x, y)$. As in the one-dimensional case, the proposed method with high order WENO reconstructions obtains the expected convergence rates.

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Division of Creative Integrated General Studies
Daegu University College
Gyeongsan 712-714, Republic of Korea
E-mail: dkyi@daegu.ac.kr

**
Division of Mechanical and Automotive Engineering
Daegu University College
Gyeongsan 712-714, Republic of Korea
E-mail: Kim.HS@daegu.ac.kr